JOURNAL OF ALGEBRA **206,** 722–727 (1998)
ARTICLE NO. JA987430

Numbers of Classes and Chains of Subgroups In Finite Groups

I. M. Isaacs

Department of Mathematics, University of Wisconsin, 480 Lincoln Drive

Communicated by Walter Feit

Received January 13, 1998

In a recent paper [4], and using relatively sophisticated representation-theoretic techniques, G. Robinson established a formula that expresses the number of classes $k(G)$ of a finite group G in terms of the local structure of the group. In two subsequent papers [2, 3], B. Külshammer and G. Robinson obtained analogous formulas for the number of p-regular classes of G, where p is a prime number, and more generally, for the number of π-classes of G, where π is an arbitrary set of prime numbers. The main purpose of this brief note is to show that each of these formulas is a special case of a much more general result that has a nearly trivial proof.

We begin with a brief review of some of the notation of Robinson and Külshammer. Let $\mathscr{F}$ denote a conjugation-invariant collection of subgroups of G, and let $\mathscr{S} = \mathscr{S}(\mathscr{F})$ denote the set of chains (linearly ordered subsets) of $\mathscr{F}$. In fact, we shall be interested only in nonempty chains, and we write $\mathscr{S}^{\#}$ to denote the set of nonempty chains of $\mathscr{F}$. (We stress that our definition of $\mathscr{S}^{\#}$ depends on the collection $\mathscr{F}$.) Of course, G acts by conjugation on $\mathscr{S}^{\#}$, and we write $\mathscr{S}^{\#}/G$ to denote an arbitrary set of representatives for the orbits of this action. For $\sigma \in \mathscr{S}^{\#}$, we write G_σ to denote the stabilizer in G of σ, so that G_σ is exactly the intersection of the normalizers of the members of σ, and $|G : G_\sigma|$ is the size of the G-orbit containing σ. Finally, we write V_σ to denote the minimum member of $\sigma \in \mathscr{S}^{\#}$.

To state Robinson's result in [4], we need a bit more notation. If $V \subseteq G$ and $N \subseteq \mathbf{N}_G(V)$, we write $k_N(V)$ to denote the number of orbits in the conjugation action of N on the elements of V. (Thus, for example,

0021-8693/98 $25.00

$k_G(G) = k(G)$.) The formula of [4] applies when $\mathscr{F}$ is the collection of all nontrivial solvable subgroups of G and also when it is the collection of those solvable subgroups that can be expressed as intersections of the maximal solvable subgroups of G. In each of these cases, Robinson shows that

$$k(G) - 1 = \sum_{\sigma \in \mathscr{S}^{\#}/G} (-1)^{1+|\sigma|}\bigl(k_{G_\sigma}(V_\sigma) - 1\bigr).$$

The formula of Külshammer and Robinson in Theorem 1 of [2] is similar. On the left side, $l(G)$ replaces $k(G)$, and on the right, $l_{G_\sigma}(V_\sigma)$ replaces $k_{G_\sigma}(V_\sigma)$. (Here, $l_N(V)$ denotes the number of orbits in the conjugation action of N on the p-regular elements of V, and $l(G) = l_G(G)$ is the total number of p-regular classes of G, where p is any prime number.) The subgroup families $\mathscr{F}$ allowed in Theorem 1 of [2] may contain some not necessarily solvable subgroups of G, but they are required to contain all nontrivial solvable subgroups and to satisfy certain other conditions.

In Theorem 2 of [3], a similar formula is given for the number of classes of π-elements of G, where π is an arbitrary set of primes. For that result, the family $\mathscr{F}$ of subgroups is taken to be all nonidentity solvable π-subgroups of G. (It should be noted that the sign in the formula of [3] is $(-1)^{|\sigma|}$ rather than $(-1)^{1+|\sigma|}$, but that difference is a consequence of a minor change in the definition of the relevant chains, and it has no real significance.)

In all of these formulas, we observe that the choice of the set $\mathscr{S}^{\#}/G$ of orbit representatives is irrelevant. We prefer to use a more symmetric formulation of these results, and so we sum over *all* chains in $\mathscr{S}^{\#}$, and we divide each summand by the size $|G:G_\sigma|$ of the corresponding orbit. Expressed in this way, Robinson's formula becomes

$$k(G) - 1 = \sum_{\sigma \in \mathscr{S}^{\#}} (-1)^{1+|\sigma|} \frac{k_{G_\sigma}(V_\sigma) - 1}{|G:G_\sigma|},$$

and similar versions exist for the formulas in [2, 3]. In fact, the above formulation is identical to the one that appeared in an earlier version of Robinson's paper.

In order to state our general result, we fix an arbitrary normal subset X of an arbitrary finite group G. If $V \subseteq G$ and $N \subseteq \mathbf{N}_G(V)$, we let $m_N(V)$ denote the number of orbits in the conjugation action of N on the elements of $V \cap X$, and we write $m(G) = m_G(G)$. Thus $m(G)$ is just the number of classes of G contained in X and $m_N(V) = 0$ precisely when $V \cap X$ is empty. Of course, the definition of $m_N(V)$ depends on our choice of the subset X. For example, if we take X to be either the set of all nonidentity elements of G or the set of all nonidentity p-regular elements of G, then $m_N(V)$ is $k_N(V) - 1$ or $l_N(V) - 1$, respectively.

THEOREM 1. *Let X be an arbitrary normal subset of a finite group G and let $\mathscr{F}$ be any conjugation-invariant collection of subgroups of G such that for each element $x \in X$, there is a unique smallest member $F_x \in \mathscr{F}$ such that $x \in F_x$. As usual, let $\mathscr{S}^\#$ be the collection of nonempty chains of $\mathscr{F}$. Then*

$$m(G) = \sum_{\sigma \in \mathscr{S}^\#} (-1)^{1+|\sigma|} \frac{m_{G_\sigma}(V_\sigma)}{|G : G_\sigma|}.$$

It is clear that if we take X to be the set of all nonidentity elements of G, then we can take $\mathscr{F}$ to be the collection of all nontrivial solvable subgroups of G, and we have $F_x = \langle x \rangle$. We thus recover most of Robinson's result in [4]. For any choice of X, we see that $\mathscr{F}$ can be an arbitrary G-invariant collection of subgroups that is closed under taking intersections and such that $X \subseteq \bigcup \mathscr{F}$. In particular, $\mathscr{F}$ can be the collection of all intersections of maximal solvable subgroups of G, and we thus recover the rest of Robinson's theorem. If we take X to be the set of nonidentity π-elements of G, we recover the formulas of Theorem 1 of [2] and Theorem 2 of [3]. But we also get results that are not included in the papers of Robinson and Külshammer. For example, we could take $\mathscr{F}$ to be the collection of cyclic groups $\langle x \rangle$ for $x \in X$, or any larger G-invariant collection of subgroups. (We mention that the paper [2] also includes other results that are not included in our Theorem 1.)

Proof of Theorem 1. By the standard formula (often wrongly attributed to Burnside) for counting orbits in a group action, we know that $|N| m_N(V)$ is equal to $\sum_{n \in N} c(n)$, where we have written $c(n)$ to denote the number of elements in $V \cap X$ that commute with n. Equivalently, we can write $|N| m_N(V) = |\{(n, x) \mid n \in N, \quad x \in V \cap X \quad \text{and} \quad nx = xn\}|$. If we write $\mathscr{P}(N, V)$ to denote this set of ordered pairs, we see that we can rewrite the formula of the theorem as

$$|\mathscr{P}(G, G)| = \sum_{\sigma \in \mathscr{S}^\#} (-1)^{1+|\sigma|} |\mathscr{P}(G_\sigma, V_\sigma)|.$$

To prove this, we observe that each of the sets $\mathscr{P}(G_\sigma, V_\sigma)$ is contained in $\mathscr{P}(G, G)$. We can thus view the right side as a "weighted sum" over all pairs $(g, x) \in \mathscr{P}(G, G)$, where the weight $w(g, x)$ attached to the pair (g, x) is the sum of $(-1)^{1+|\sigma|}$ over all $\sigma \in \mathscr{S}^\#$ for which (g, x) lies in $\mathscr{P}(G_\sigma, V_\sigma)$. Since the left side of the desired equation is just the number of pairs (g, x) in $\mathscr{P}(G, G)$, we see that it suffices to show that the weight $w(g, x) = 1$ for each pair $(g, x) \in \mathscr{P}(G, G)$.

To compute $w(g, x)$, we need to determine all chains $\sigma \in \mathscr{S}^{\#}$ for which $(g, x) \in \mathscr{P}(G_\sigma, V_\sigma)$. The relevant chains are exactly those chains σ for which $g \in G_\sigma$ and $x \in V_\sigma$. That $x \in V_\sigma$ is, of course, equivalent to saying that x is in every member of the chain σ, and the condition that $g \in G_\sigma$ says that every member of the chain is invariant under g. The chains $\sigma \in \mathscr{S}^{\#}$ relevant to the pair (g, x), therefore, are exactly the nonempty chains in the poset we shall call $\mathscr{L} = \mathscr{L}(g, x)$, consisting of all g-invariant members of $\mathscr{F}$ that contain the element x. This shows that

$$w(g, x) = \sum_{\sigma \in \mathscr{S}(\mathscr{L})^{\#}} (-1)^{1+|\sigma|}.$$

We observe that g centralizes x since $(g, x) \in \mathscr{P}(G, G)$, and thus g stabilizes the unique minimum member F_x of $\mathscr{F}$ that contains x. It follows that $\mathscr{L}(g, x)$ has a unique minimum member. The desired conclusion that $w(g, x) = 1$ for all pairs $(g, x) \in \mathscr{P}(G, G)$ is thus immediate from the following easy lemma. ∎

LEMMA 2. *Let $\mathscr{L}$ be a finite poset containing a unique minimum member m, and write $\mathscr{S}(\mathscr{L})^{\#}$ to denote the collection of nonempty chains of $\mathscr{L}$. Then*

$$\sum_{\sigma \in \mathscr{S}(\mathscr{L})^{\#}} (-1)^{1+|\sigma|} = 1.$$

Proof. Given any member $\sigma \in \mathscr{S}(\mathscr{L})^{\#}$ that does not contain m, we see that $\sigma \cup \{m\}$ is another member of $\mathscr{S}(\mathscr{L})^{\#}$ whose contribution to the sum exactly cancels that of σ. The only member of $\mathscr{S}(\mathscr{L})^{\#}$ not yet accounted for is the singleton chain $\{m\}$. Since its contribution to the sum is 1, the result follows. ∎

Next, we generalize and simplify Theorem 1 of [3], which provides an alternative formula for the number of π-classes of G.

THEOREM 3. *Let X be a normal subset of a finite group G, and let $\mathscr{F}$ be a conjugation-invariant family of subgroups of G having the property that for each element $x \in X$ there is a unique minimum member F_x of $\mathscr{F}$ that contains x. Assume in addition that if x normalizes $F \in \mathscr{F}$, then the subgroup $\langle F_x, F \rangle$ lies in $\mathscr{F}$. For subgroups $H \subseteq G$, write $m(H)$ to denote the number of conjugacy classes of H that are contained in X, and as usual, let $\mathscr{S}^{\#}$ be the set of nonempty chains of $\mathscr{F}$. Then*

$$m(G) = \sum_{\sigma \in \mathscr{S}^{\#}} (-1)^{1+|\sigma|} \frac{m(G_\sigma)}{|G : G_\sigma|}.$$

Theorem 1 of [3] is essentially the case of our Theorem 3 in the case where X is the set of π-elements of G and $\mathscr{F}$ is the family of nontrivial solvable π-subgroups of G. In this situation, we see that $F_x = \langle x \rangle$ for $x \in X$, and thus if x normalizes $F \in \mathscr{F}$, we have $\langle F_x, F \rangle = F\langle x \rangle$, and this lies in $\mathscr{F}$, as required.

The proof of Theorem 3 is similar to that of Theorem 1, except that we need to replace Lemma 2 by a more technical and more general result about chains in posets. To state this generalization, we remind the reader that an element a in a poset $\mathscr{L}$ is *conjunctive* (according to Definition 2.6 of [1]) if for every element $x \in \mathscr{L}$, there is a unique smallest element $y \in \mathscr{L}$ such that both $y \geq x$ and $y \geq a$. (We write $y = a \vee x$.) In particular, if $\mathscr{L}$ has a unique minimal element m, then m is conjunctive in $\mathscr{L}$.

LEMMA 4. *Let $\mathscr{L}$ be a finite poset that contains a conjunctive element, and write $\mathscr{S}(\mathscr{L})^{\#}$ to denote the collection of nonempty chains of $\mathscr{L}$. Then*

$$\sum_{\sigma \in \mathscr{S}(\mathscr{L})^{\#}} (-1)^{1+|\sigma|} = 1.$$

Proof. Create a new poset $\mathscr{P}$ by adjoining an element 0 to $\mathscr{L}$, defined to be smaller than every element in $\mathscr{L}$. Let $a \in \mathscr{L}$ be conjunctive in $\mathscr{L}$ and note that a is still conjunctive when it is viewed as an element of $\mathscr{P}$. For elements $y \in \mathscr{P}$, write $\mu(y) = \mu(0, y)$ to denote the Möbius function associated with the poset $\mathscr{P}$. Also, for $\sigma \in \mathscr{S}(\mathscr{L})^{\#}$, we write $M(\sigma)$ for the maximum element of the chain σ. If $y \in \mathscr{L}$, therefore, we know by Lemma 2.2 of [1] that $\mu(y) = \sum(-1)^{|\sigma|}$, where the sum is over all chains $\sigma \in \mathscr{S}(\mathscr{L})^{\#}$ such that $M(\sigma) = y$.

We now have

$$1 + \sum_{\sigma \in \mathscr{S}(\mathscr{L})^{\#}} (-1)^{|\sigma|} = \sum_{y \in \mathscr{P}} \mu(y) = \sum_{b \in \mathscr{L}} \left(\sum_{\substack{y \in \mathscr{P} \\ y \vee a = b}} \mu(y) \right) = 0,$$

where the last equality holds because the inner sum vanishes by Lemma 2.7 of [1]. The result now follows. ∎

Proof of Theorem 3. We need to show that

$$|G| m(G) = \sum_{\sigma \in \mathscr{S}^{\#}} |G_\sigma| m(G_\sigma).$$

The left side of this equation is the total number of pairs of elements (g, x), where $g \in G$, $x \in X$, and $xg = gx$. As in the proof of Theorem 1, the right side can be viewed as a weighted sum over all such pairs, where the weight $w(g, x)$ associated with a pair is the sum of the quantity $(-1)^{1+|\sigma|}$ over all chains $\sigma \in \mathscr{S}^{\#}$ that are "relevant" to the pair (g, x). The relevant chains in this situation are exactly the nonempty chains in the poset $\mathscr{L}$ consisting of all members of $\mathscr{F}$ that are normalized by both x and g. Now F_x lies in $\mathscr{L}$, and in fact F_x is a conjunctive element of $\mathscr{L}$ because if $F \in \mathscr{L}$, we see that x normalizes F, and thus $F \vee F_x = \langle F, F_x \rangle$ lies in $\mathscr{L}$. It follows by Lemma 4 that $w(g, x) = 1$, and the proof is complete. ∎

Before we close, we mention that there are in fact a number of other elementary formulas that can be used to count the number of classes of a group using only local information. As just one example, we present the following, which does not involve chains of subgroups.

THEOREM 5. *Let $\mathscr{F}$ be the family of all nontrivial cyclic subgroups of a finite group G, and for $C \in \mathscr{F}$, write $s(C)$ to denote the number of classes of elements of $\mathbf{N}_G(C)$ that contain a generator for C. Then*

$$k(G) - 1 = \sum_{C \in \mathscr{F}} \frac{s(C)}{|G : \mathbf{N}_G(C)|}.$$

Proof. It suffices to prove that

$$|G|(k(G) - 1) = \sum_{C \in \mathscr{F}} |\mathbf{N}_G(C)| s(C).$$

The left side of this equation is equal to the number of pairs (g, x), such that g and x are commuting elements of G and $x \neq 1$. On the other hand, for each nontrivial cyclic subgroup C, we see that $|\mathbf{N}_G(C)| s(C)$ is equal to the number of such pairs (g, x) where $\langle x \rangle = C$. (Note that if $\langle x \rangle = C$, then since g and x commute, we necessarily have $g \in \mathbf{N}_G(C)$.) Equality thus follows, and the proof is complete. ∎

REFERENCES

1. T. Hawkes, I. M. Isaacs, and M. Özaydın, On the Möbius function of a finite group, *Rocky Mountain J. Math.* **19** (1989), 1003–1034.
2. B. Külshammer and G. R. Robinson, On Alperin's conjecture and certain subgroup complexes, *Algebras Representation Theory*, in press.
3. B. Külshammer and G. R. Robinson, On the number of conjugacy classes of π-elements, *J. Algebra*, in press.
4. G. R. Robinson, On local control of the number of conjugacy classes, *J. Algebra*, in press.